DSP

Chapter-5 : Filter Implementation

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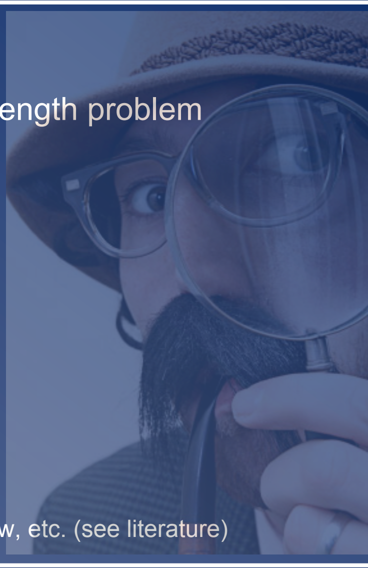
Filter Design Process

- **Step-1 : Define filter specs**
  Pass-band, stop-band, optimization criterion,…
- **Step-2 : Derive optimal transfer function**
  FIR or IIR design **Chapter-3**
- **Step-3 : Filter realization** (block scheme/flow graph)
  Direct form realizations, lattice realizations,… **Chapter-4**
- **Step-4 : Filter implementation** (software/hardware)
  Finite word-length issues, … **Chapter-5**
  Question: implemented filter = designed filter ?
  'You can't always get what you want' -Jagger/Richards (?)

Chapter-5 : Filter Implementation



- **Introduction**
  Filter implementation & finite wordlength problem
- **Coefficient Quantization**
- **Arithmetic Operations**
  Quantization noise
  Statistical Analysis
  Limit Cycles
  Scaling

- **PS: Short version, does not include…**
  Fixed & floating point representations, overflow, etc. (see literature)

Introduction

Back to Chapter-4…

Q:Why bother

about many different realizations
for one and the same filter?

Introduction

Filter implementation/finite word-length problem

- So far have assumed that signals/coefficients/arithmetic operations are represented/performed with <u>infinite</u> precision
- In practice, numbers can be represented only to a <u>finite</u> precision, and hence signals/coefficients/arithmetic operations are subject to quantization (truncation/ rounding/...) errors

- Investigate impact of…
  - **quantization of filter coefficients**
  - **quantization in arithmetic operations**

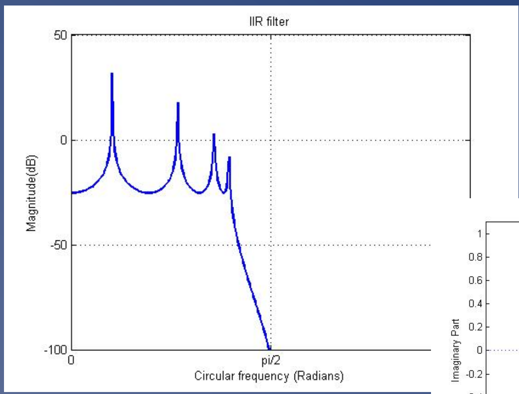
Introduction

Filter implementation/finite word-length problem

- We consider <u>fixed-point</u> filter implementations, with a `short' word-length
  In hardware design, with tight speed requirements, finite word-length problem is a relevant problem

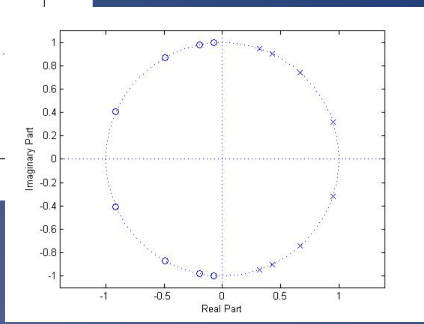
- In signal processors with a `sufficiently long ' word-length, e.g. with 24 bits (=7 decimal digits) precision, or with <u>floating-point</u> representations and arithmetic, finite word-length issues are less relevant

Introduction: Example



Transfer function

- % IIR Elliptic Lowpass filter designed using
- % ELLIP function.
- % All frequency values are in Hz.
- Fs = 48000; % Sampling Frequency
- L = 8; % Order
- Fpass = 9600; % Passband Frequency
- Apass = 60; % Passband Ripple (dB)
- Astop = 160; % Stopband Attenuation (dB)
-



Poles & zeros

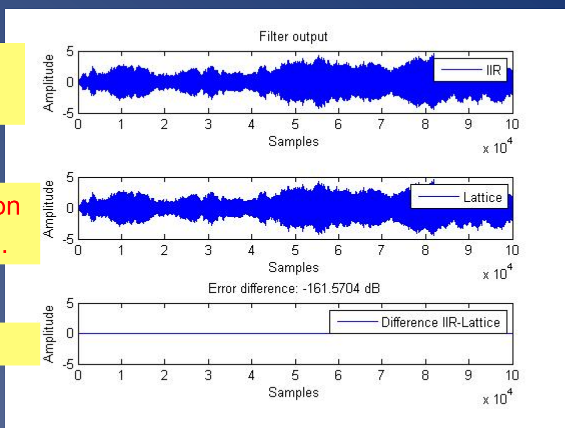
Introduction: Example

Filter outputs…

Direct form realization @ infinite precision…

Lattice-ladder realization @ infinite precision…

Difference…



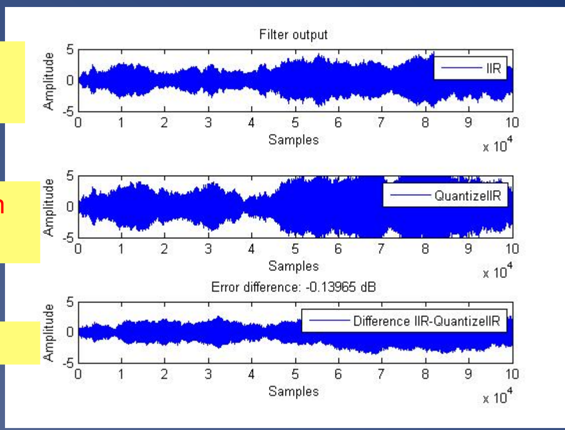
Introduction: Example

Filter outputs…

Direct form realization @ infinite precision…

Direct form realization @ 8-bit precision…

Difference…



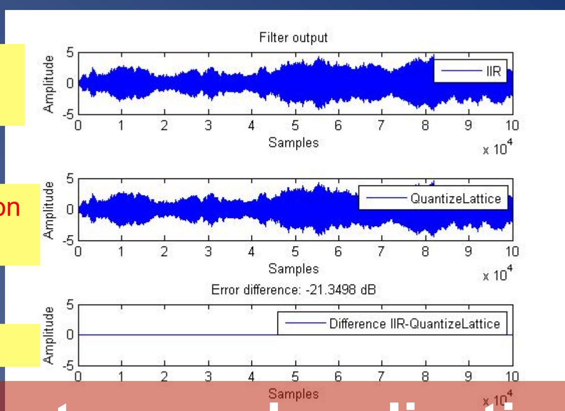
Introduction: Example

Filter outputs…

Direct form realization @ infinite precision…

Lattice-ladder realization @ 8-bit precision…

Difference…



Better select a good realization !

Coefficient Quantization

Coefficient quantization problem

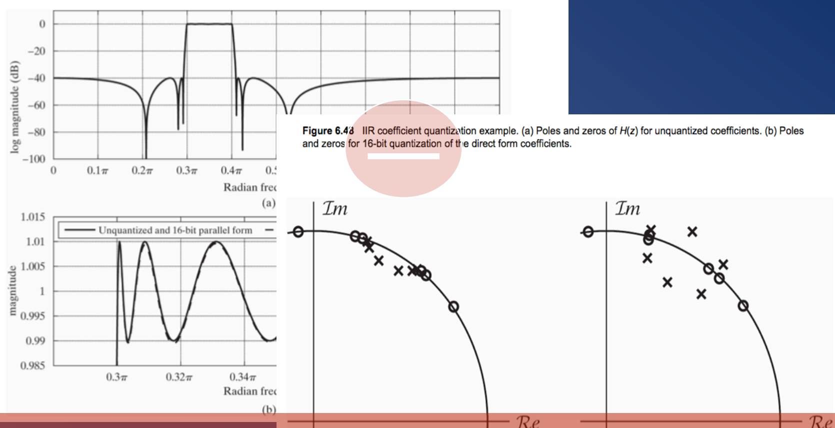
- Filter design in Matlab (e.g.) provides filter coefficients to 15 decimal digits (such that filter meets specifications)
- For implementation, have to quantize coefficients to the word-length used for the implementation
- As a result, implemented filter may fail to meet specifications…

Example from

PEARSON *Discrete-Time Signal Processing*, Third Edition
Alan V. Oppenheim • Ronald W. Schafer

Figure 6.47 IIR coefficient quantization example. (a) Log magnitude for unquantized elliptic bandpass filter. (b) Magnitude in passband for unquantized (solid line) and 16-bit quantized cascade form (dashed line).

Figure 6.48 IIR coefficient quantization example. (a) Poles and zeros of *H*(*z*) for unquantized coefficients. (b) Poles and zeros for 16-bit quantization of the direct form coefficients.



Better select a good realization !

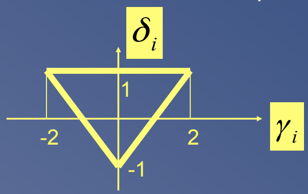
Coefficient Quantization

Coefficient quantization effect on pole locations

- Example : 2nd-order system (e.g. for cascade/direct form realization)

$$H_i(z) = \frac{1+\alpha_i.z^{-1}+\beta_i.z^{-2}}{1+\gamma_i.z^{-1}+\delta_i.z^{-2}}$$

`Triangle of stability' : denominator polynomial is stable (i.e. roots inside unit circle) iff coefficients lie inside triangle…



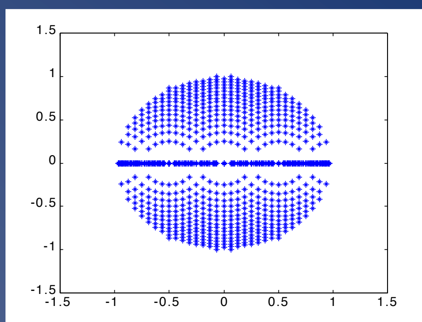
Proof: Apply Schur-Cohn stability test (see Chapter-4).

Coefficient Quantization

- Example (continued)

With 5 bits per coefficient, all possible `quantized' pole positions are...

for $\gamma_i = -2 : 0.1250 : 2$
  for $\delta_i = -1 : 0.0625 : 1$
    plot(poles) (if stable)
  end
end



Low density of `quantized' pole locations at z=1, z=-1,
hence problem for narrow-band LP and HP filters in (transposed) direct form (see Chapter-3).

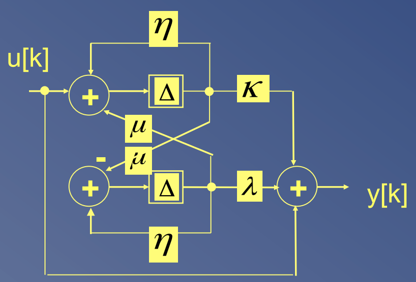
Coefficient Quantization

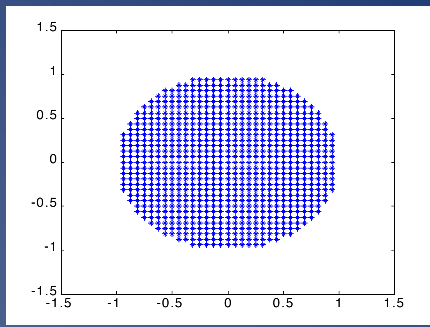
- Example (continued)

  Possible remedy: `coupled realization'

  Poles are $\eta \pm j.\mu$ where $-1 < \eta, \mu < 1$ are realized/quantized

  hence 'quantized' pole locations are (5 bits)





coefficient precision = pole precision

Coefficient Quantization

Coefficient quantization effect on pole locations

- Higher-order systems (first-order analysis)

polynomial : $1 + a_1.z^{-1} + a_2.z^{-2} + \ldots + a_L.z^{-L}$

roots are : $p_1, p_2, \ldots, p_L$

`quantized' polynomial: $1 + \hat{a}_1.z^{-1} + \hat{a}_2.z^{-2} + \ldots + \hat{a}_L.z^{-L}$

`quantized' roots are: $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_L$

$$\hat{p}_l - p_l \approx -\sum_{i=1}^{L} \frac{p_l^{L-i}}{\prod_{j \neq l} (p_l - p_j)} .(\hat{a}_i - a_i)$$

➔ Tightly spaced poles (e.g. for narrow band filters) imply high sensitivity of pole locations to coefficient quantization

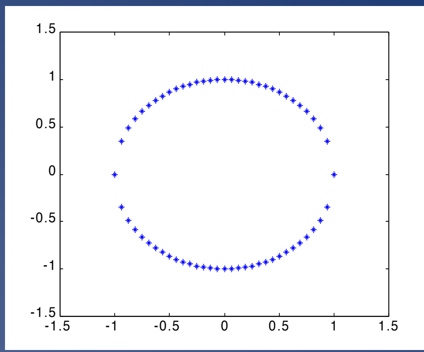
➔ Hence preference for low-order systems (e.g. in parallel/cascade)

Coefficient Quantization

Coefficient quantization effect on zero locations

- Analog filter design + bilinear transformation often lead to numerator polynomial of the form (e.g. 2nd-order cascade realization)

$1-2\cos\theta_i . z^{-1}+z^{-2}$ hence with zeros always on the unit circle



Quantization of the coefficient $2\cos\theta_i$ shifts zeros on the unit circle, which mostly has only minor effect on the filter characteristic.

Hence mostly ignored…

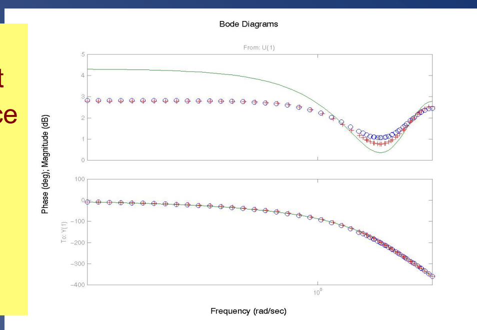
Coefficient Quantization

Coefficient quantization in lossless lattice realizations

o = original transfer function

+ = transfer function after 8-bit truncation of lossless lattice filter coefficients

- = transfer function after 8-bit truncation of direct-form coefficients (bi’s)



In lossless lattice, all coefficients are sines and cosines, hence all values between –1 and +1…, i.e. `dynamic range’ and coefficient quantization error well under control.

Arithmetic Operations

Quantization noise problem

- If two B-bit numbers are added, the result is a B+1 bit number.
- If two B-bit numbers are multiplied, the result is a 2B-1 bit number.
- Typically (especially so in an IIR (feedback) filter), the result of an addition/multiplication has to be represented again as a B'-bit number (e.g. B'=B). Hence have to remove least significant bits (*).
- Rounding/truncation/… to B' bits introduces **quantization noise**.

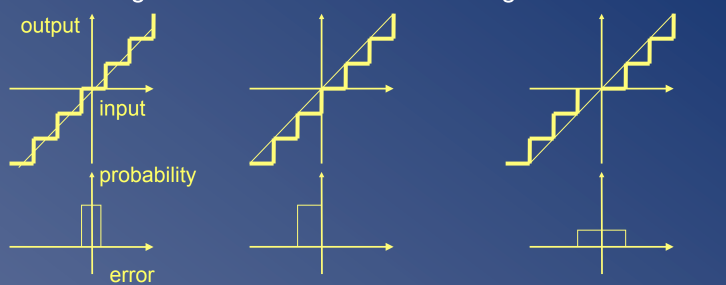
- The effect of quantization noise is usually analyzed in a statistical manner (see p.20-25)
- Quantization, however, is a deterministic non-linear effect, which may give rise to **limit cycle oscillations** (see p.26-30)

(*) ..and/or most significant bits - not considered here

Quantization Noise / Statistical Analysis

Quantization mechanisms

Rounding | Truncation | Magnitude Truncation



Rounding	Truncation	Magnitude Truncation
mean=0	mean=(-0.5)LSB (biased!)	mean=0
variance=(1/12)LSB^2	variance=(1/12)LSB^2	variance=(1/6)LSB^2

PS: …assuming input to quantization is uniformly distributed (is it?)

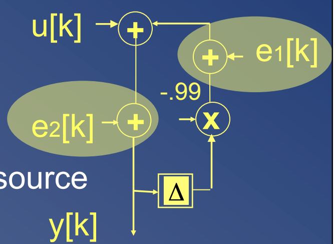
Quantization Noise / Statistical Analysis

Statistical analysis is based on the following **assumptions** :

- Each quantization error is random, i.e. uncorrelated/independent of the number that is quantized, and with uniform probability distribution function (see previous slide) (ps: model more suited for multipliers than for adders)
- Successive quantization errors at the output of a given multiplier/adder are uncorrelated/independent (=white noise assumption)
- Quantization errors at the outputs of different multipliers/adders are uncorrelated/independent (=independent sources assumption)

➔One noise source is inserted after each **multiplier/adder**

➔Since the filter is a **linear filter** the output noise generated by each noise source is added to the output signal



Quantization Noise / Statistical Analysis

Effect on the output signal of a noise generated at a particular point in the filter is computed as follows:

- Noise is e[k], assumed white (=flat PSD) with mean & variance μ_e, σ_e^2
- Transfer function from from e[k] to filter output is G(z),g[k] (= 'noise transfer function')
- Noise mean at the output is $\mu_e.(\text{'DC} - \text{gain'}) = \mu_e.G(z)|_{z=1}$
- Noise variance at the output is

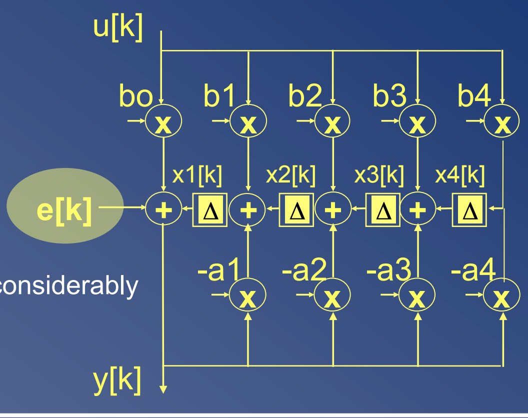
$$\sigma_e^2.(\text{'noise - gain'}) = \sigma_e^2.(\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(e^{j\omega})|^2 d\omega)$$

$$= \sigma_e^2.\sum_{k=0}^{\infty}|g[k]|^2 = \sigma_e^2.\|g\|_2^2$$

Repeat procedure for each noise source…

Quantization Noise / Statistical Analysis

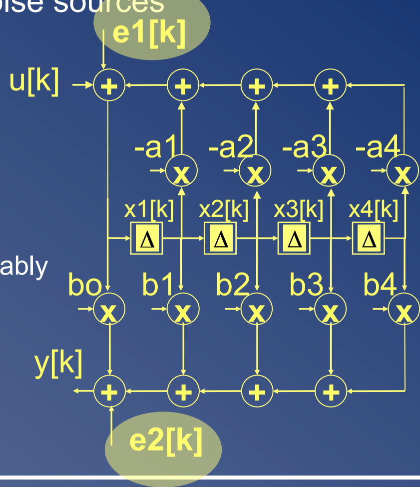
PS: In a <u>transposed direct form</u> realization all noise transfer functions are equal (up to delay), hence all noise sources can be lumped into <u>one</u> equivalent noise source



…which simplifies analysis considerably

Quantization Noise / Statistical Analysis

PS: In a <u>direct form</u> realization all noise sources can be lumped into <u>two</u> equivalent noise sources



…which simplifies analysis considerably

Quantization Noise / Statistical Analysis

PS: Quantization noise of ***A/D-converters*** can be modeled/ analyzed in a similar fashion.

Noise transfer function is filter transfer function H(z)

PS: Quantization noise of ***D/A-converters*** can be modeled/ analyzed in a similar fashion.

Non-zero quantization noise if D/A converter wordlength is shorter than filter wordlength.

Noise transfer function = 1

Quantization Noise / Limit Cycles

Statistical analysis is simple/convenient, but quantization is truly a **non-linear** effect, and should be analyzed as a **deterministic** process

Though very difficult, such analysis may reveal odd behavior :

Example: y[k] = -0.625.y[k-1]+u[k]
4-bit rounding arithmetic
input u[k]=0, y[0]=3/8
output y[k] = 3/8, -1/4, 1/8, -1/8, 1/8, -1/8, 1/8, -1/8, 1/8,..

Oscillations in the absence of input (u[k]=0) are called

`zero-input limit cycle oscillations'

Quantization Noise / Limit Cycles

Example: y[k] = -0.625.y[k-1]+u[k]
4-bit truncation (instead of rounding)
input u[k]=0, y[0]=3/8
output y[k] = 3/8, -1/4, 1/8, 0, 0, 0,.. (no limit cycle!)

Example: y[k] = 0.625.y[k-1]+u[k]
4-bit rounding
input u[k]=0, y[0]=3/8
output y[k] = 3/8, 1/4, 1/8, 1/8, 1/8, 1/8,..

Example: y[k] = 0.625.y[k-1]+u[k]
4-bit truncation
input u[k]=0, y[0]=-3/8
output y[k] = -3/8, -1/4, -1/8, -1/8, -1/8, -1/8,..

Conclusion: weird, weird, weird,… !

Quantization Noise / Limit Cycles

- Limit cycle oscillations are clearly **unwanted** (e.g. may be audible in speech/audio applications)
- Limit cycle oscillations can only appear if the filter has feedback. Hence **FIR filters** cannot have limit cycle oscillations
- Mathematical analysis is very difficult ☹

Quantization Noise / Limit Cycles

- Truncation often helps to avoid limit cycles (e.g. **magnitude truncation**, where absolute value of quantizer output is never larger than absolute value of quantizer input (=`passive quantizer'))
- Some filter realizations can be made limit cycle free, e.g. coupled realization, orthogonal filters (details omitted)

Quantization Noise / Limit Cycles

Here's the good news:

For a..

- lossless lattice realization of a general IIR filter
- lattice-ladder realization of a general IIR filter

…and when magnitude truncation (=`passive quantization') is used, the filter is **guaranteed to be free of limit cycles**!

(details omitted)

Intuition: quantization consumes energy/power, orthogonal filter operations do not generate power to feed limit cycle

Scaling

The scaling problem

- Finite word-length implementation implies maximum representable number. Whenever a signal (output or internal) exceeds this value, ***overflow*** occurs.
- Digital overflow may lead (e.g. in 2's-complement arithmetic) to polarity reversal (instead of saturation such as in analog circuits), hence may be very harmful.
- Avoid overflow through proper signal ***scaling***, implemented by bit shift-operations applied to signals, or by scaling of filter coefficients, or..
- Scaled transfer function may be c.H(z) instead of H(z) (hence need proper tracing of scaling factors)

Scaling

The scaling problem

Further details see…

- Literature
- http://homes.esat.kuleuven.be/~dspuser/DSP-CIS/2016-2017/material.html